Products of Monoids and Its Applications on the Monoids of State Machines

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Abstract

Let M be a monoid and X a non-empty set. M will be called a transformation monoid on X if there is a mapping $\phi : M \times X \longrightarrow X$, for which we write $\phi(m, x) = m \cdot x$ and which satisfies the conditions:

1) $(m_1 m_2) \cdot x = m_1 \cdot (m_2 \cdot x)$, for each $x \in X$ and for each $m_1, m_2 \in M$.
2) $1_M \cdot x = x$, for each $x \in X$.

Let M and N be two monoids. Let N^M be the set of all functions defined on M with values in N. In this paper, we prove that the set N^M forms a monoid shch that for any $\varphi, \psi \in N^M$, let $\varphi\psi \in N^M$ in N^M be defined for all $m \in M$ by: $(\varphi\psi)(m) = \varphi(m)\psi(m)$, the monoid M is a transformation monoid on N^M in the following was:
if $m \in M, \varphi \in N^M$, then $(m \cdot \varphi)(x) = \varphi^m(x) = \varphi(xm)$ for $x \in M$, and the set of all pairs (m, φ) where $m \in M, \varphi \in N^M$, with multiplications operation given by: $(m, \varphi)(m', \psi) = (mm', \varphi\psi^m)$ where $m, m' \in M$ and $\varphi, \psi \in N^M$ is a monoid. On the other hand, we present the direct product, the cascade product and wreath product of state machines, also we calculate the monoids of state machines.

Keywords: Free Monoid; Monoid of State Machine; Morphism of Monoids; State Machine; Transformation Monoid; Wreath Product of Monoids

1 Introduction

The theory of machines that has developed in last twenty years, has had a considerable influence, not only on the computer systems, but also biology, biochemestry, *etc.*

A semigroup is an ordered pair $(S, \cdot)$, where S is non -empty set and the dot " $\cdot$" is a binary operation on S, i.e., a mapping $(a, b) \longmapsto a \cdot b$ from $S \times S$ to S such that for all $a, b, c \in S, (a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative law).

A semigroup $(S, \cdot)$ with the identity element is called a monoid [10].

A state machine is a 3-tuple (Q, Σ, δ), where

1) Q is a finite nonempty set (the set of states);
2) Σ is a finite alphabet (the set of inputs);
3) δ is a function of $Q \times \Sigma$ into Q (the transition function) [6, 7].

The remainder of this paper is organized as follows. In Section 2, we begin with some elementary material concerning of monoids and state machines. In Section 3, we present direct product, semidirect product and wreath product of monoids. In Section 4, we introduce the direct product of state machines, the cascade product and wreath product, also we calculate the monoids of state machines. Finally, we draw our conclusions in Section 5.

2 Preliminaries

A monoid $(M, \cdot)$ consists of a set M together with a binary operation " $\cdot$ " on M such that

1) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in M$;

2) There existes an identity $1_M \in M$ such that

$$a \cdot 1_M = 1_M \cdot a = a \text{ for all } a \in M.$$

A monoid $(M, \cdot)$ is called commutative if the operation " $\cdot$ " is commutative. Hence a semigroup $(S, \cdot)$ is just a set S together with an associative binary operation.

Let X be any set and let $X^X = \{f : X \longrightarrow X\}$ be the set of all function from X to itself. Then $\left(X^X, \circ\right)$ is a monoid, called the transformation monoide of X. In fact, the analogue of Cayley's theorem holds for monoids, and it can be shown that every monoid can be represented as a transformation monoid.

Suppose that $\mathcal{R}$ is an equivalence relation on a monoid $(M, \cdot)$. Then $\mathcal{R}$ is called a congruence relation on $(M, \cdot)$ if $a\mathcal{R}b$ implies $ac\mathcal{R}bc$ and $ca\mathcal{R}cb$ for all $a, b, c \in M$. The congruence class containing the element $m \in M$ is the set $[m] = \{x \in M : x\mathcal{R}m\}$.

If $\mathcal{R}$ is a congruence relation on the monoid $(M, \cdot)$, the quotient set $M/\mathcal{R} = \{[m] : m \in M\}$ is a monoid under the operation defined by $[m]\,[n] = [mn]$. This monoid is called the quotient monoid of M by $\mathcal{R}$.

If $(M, \cdot)$ and (N, Δ) are two monoids, with identities 1_M and 1_N, respectively, then the function $f : M \longrightarrow N$ is a monoid morphism from $(M, \cdot)$ to (N, Δ) if

1) $f\,(x \cdot y) = f\,(x)\,\Delta f\,(y)$ for all $x, y \in M$,

2) $f\,(1_M) = 1_N$.

A monoid isomorphism is simply a bijective monoid morphism.

Let M be a monoid and X a non-empty set. M will be called a transformation monoid on X if there is a mapping $\phi : M \times X \longrightarrow X$, for which we write $\phi\,(m, x) = m \cdot x$ and which satisfies the conditions:

1) $(m_1 m_2) \cdot x = m_1 \cdot (m_2 \cdot x)$, for each $x \in X$ and for each $m_1, m_2 \in M$.

2) $1_M \cdot x = x$, for each $x \in X$.

For every transformation monoid M on X, there is a homomorphism $\psi : M \longrightarrow E\,(X)$, the monoid of all mappings $f : X \longrightarrow X$, given by $\psi\,(m) = f$, where $f\,(x) = m \cdot x$ for all $x \in X$.

We formally define an alphabet as a non-empty finite set. A word over an alphabet Σ is a finite sequence of symbols of Σ. Although one writes a sequence as $(\sigma_1, \sigma_2, \cdots, \sigma_n)$, in the present context, we prefer to write it as $\sigma_1\sigma_2 \cdots \sigma_n$. The set of all words on the alphabet Σ is denoted by Σ^* and is equipped with the associative operation defined by the concatenation of two sequences [1, 5]:

$$\alpha_1\alpha_2 \cdots \alpha_n\beta_1\beta_2 \cdots \beta_m = \alpha_1\alpha_2 \cdots \alpha_n\beta_1\beta_2 \cdots \beta_m.$$

This operation is associative. This allows us to write $w = \sigma_1\sigma_2\cdots\sigma_n$. The string consisting of zero letters is called the empty word, written ϵ. Thus, $\epsilon, \alpha, \beta, \alpha\alpha\beta\alpha, \alpha\alpha\alpha\beta\alpha$ are words over the alphabet $\{\alpha, \beta\}$. Thus the set Σ^* of words is equipped with the structure of a monoid. the monoid Σ^* is called the free monoid on Σ. The length of a word w, in symbols $|w|$, is the number of letters in w when each letter is counted as many times as it occurs. Again by definition, $|\epsilon| = 0$. For example $|\alpha\beta\alpha| = 3$ and $|\alpha\beta\alpha\beta\alpha\alpha| = 6$. Let w be a word over an alphabet Σ. For $\sigma \in \Sigma$, the number of occurrences of σ in w shall be denoted by $|w|_\sigma$. For example $|\alpha\beta\alpha|_\beta = 1$ and $|\alpha\beta\alpha\beta\alpha\alpha|_\alpha = 4$ [3, 4].

Let $(\Sigma^*, \cdot)$ be the free monoid generated by Σ and let $i : \Sigma \longrightarrow \Sigma^*$ be the function that maps each element σ of Σ into the corresponding word of length 1, so that $i(\sigma) = \sigma$. Then if $f : \Sigma \longrightarrow M$ is any function into the underlying set of any monoid $(M, \cdot)$, there is a unique monoid morphism $h : (\Sigma^*, \cdot) \longrightarrow (M, \cdot)$ such that $h \circ i = f$.

A state machine is a triple $S = (Q, \Sigma, \delta)$ where Q is a finite set of states, Σ is a finite set of symbols called the input alphabet and $\delta : Q \times \Sigma \longrightarrow Q$ is a partial function called the transition function. The transition function can be extended naturally to sequences of input symbols, by letting $\delta(q, w\sigma) = \delta(\delta(q, w), \sigma)$ and $\delta(q, \epsilon) = q$, for all $w \in \Sigma^*, \sigma \in \Sigma$ and $q \in Q$.

A state machine $S = (Q, \Sigma, \delta)$ is called complete if the partial function $\delta : Q \times \Sigma \longrightarrow Q$ is in fact a function. In this situation we can specify what the resultant $\delta(q, \sigma)$ is for all possible combinations of $q \in Q$ and $\sigma \in \Sigma$.

Let $\sigma \in \Sigma$, define $\delta_\sigma : Q \longrightarrow Q$ by $\delta_\sigma(q) = \delta(q, \sigma)$ for each $q \in Q$. Let $w \in \Sigma^+$ be a word of length at least 1 with symbols from Σ. Suppose that $w = \sigma_1\sigma_2\cdots\sigma_n$ then we define $\delta_w : Q \longrightarrow Q$ by $\delta_w(q) = \delta_{\sigma_n}\delta_{\sigma_{n-1}}\cdots\delta_{\sigma_1}(q)$.

Let $S = (Q, \Sigma, \delta)$ be a state machine, define the monoid morphism $h : (\Sigma^*, \cdot) \longrightarrow (Q^Q, \circ)$ by $h(w) = \delta_w$. Define the relation $\mathcal{R}$ on Σ^* by $w\mathcal{R}w'$ if and only if $h(w) = h(w')$. This is easily verified to be an equivalence relation. Furthermore, it is a congruence relation. The qoutient monoid $(\Sigma^*/\mathcal{R}, \cdot)$ is called the monoid of the state machine (Q, Σ, δ).

3 Direct Product, Semidirect Product and Wreath Product of Monoids

In this section, we present direct product, semidirect product and wreath product of monoids.

Proposition 1. *Let M and N be monoids, consider the set $M \times N$ the cartesian product of M and N, and define a multiplication "$\cdot$" on $M \times N$ as follows:*

$$(m_1, n_1) \cdot (m_2, n_2) = (m_1 m_2, n_1 n_2).$$

This result is a monoid $(M \times N, \cdot)$ with is called the direct product of M and N.

Proof. It is easy to show that this product is associative and the identity element in $M \times N$ is $(1_M, 1_N)$. □

Example 1. *Let $M = (\mathbb{N}, +)$ and $N = (\mathbb{N}, \times)$, then in the direct product $M \times N$ we have $(m, n) \cdot (r, s) = (m + r, n \times s)$.*

Proposition 2. *Given three monoids M, N, L we can form the direct product $(M \times N) \times L$ similarly $M \times (N \times L)$ and the relationship between these two monoids is the isomorphism [6]:*

$$(M \times N) \times L \cong M \times (N \times L).$$

Proof. The isomorphism is $h : (M \times N) \times L \longrightarrow M \times (N \times L)$ defined by

$$h\left(\left(m, n\right), l\right) = \left(m, \left(n, l\right)\right),$$

where $m \in M, n \in N, l \in L$. □

Proposition 3. *Given any monoids M and N, suppose that $\theta : N \longrightarrow End\left(M\right)$ is a monoid morphism. Then $(M \times N, \cdot)$ is a monoid under the operation " $\cdot$ " defined by $(m, n) \cdot (m', n') = \left(m\theta\left(n\right)\left(m'\right), nn'\right)$ where $m, m' \in M, n, n' \in N$. The monoid $(M \times N, \cdot)$ is called the semidirect product of M and N with respect to θ and it is denoted by $M \times_\theta N$.* [1, 8, 9]

Proof. From [6]:

1) We will prove that " $\cdot$ " is associative on $M \times N$: let $m, m', m" \in M, n, n', n'' \in N$, we have

$$\begin{aligned} & \left(\left(m, n\right) \cdot \left(m', n'\right)\right) \cdot \left(m'', n''\right) \\ = & \left(m\theta\left(n\right)\left(m'\right), nn'\right) \cdot \left(m'', n''\right) \\ = & \left(m\theta\left(n\right)\left(m'\right)\theta\left(nn'\right)\left(m''\right), \left(nn'\right)n''\right) \\ = & \left(m\theta\left(n\right)\left(m'\right)\theta\left(n\right)\left(\theta\left(n'\right)\left(m''\right)\right), \left(nn'\right)n''\right) \\ = & \left(m\theta\left(n\right)\left(m'\theta\left(n'\right)\left(m''\right)\right), \left(nn'\right)n''\right). \end{aligned}$$

Also we have

$$\begin{aligned} & \left(m, n\right) \cdot \left(\left(m', n'\right) \cdot \left(m'', n''\right)\right) \\ = & \left(m, n\right)\left(m'\theta\left(n'\right)\left(m''\right), n'n''\right) \\ = & \left(m\theta\left(n\right)\left(m'\theta\left(n'\right)\left(m''\right)\right), n\left(n'n''\right)\right). \end{aligned}$$

Then " $\cdot$ " is associative on $M \times N$.

2) For $(m, n) \in M \times N$, $(m, n) \cdot (1_M, 1_N) = \left(m\theta\left(n\right)\left(1_M\right), n1_N\right) = (m1_M, n) = (m, n)$. Also $(1_M, 1_N) \cdot (m, n) = \left(1_M\theta\left(1_N\right)\left(m\right), 1_N n\right) = \left(1_M Id_M\left(m\right), n\right) = (m, n)$. Hence $(M \times N, \cdot)$ is a monoid.

□

Proposition 4. *Let M and N be two monoids. Let N^M be the set of all functions defined on M with values in N.*

1) The set N^M forms a monoid shch that for any $\varphi, \psi \in N^M$, let $\varphi\psi \in N^M$ in N^M be defined for all $m \in M$ by: $\left(\varphi\psi\right)\left(m\right) = \varphi\left(m\right)\psi\left(m\right)$.

2) The monoid M is a transformation monoid on N^M in the following was:

- *If $m \in M, \varphi \in N^M$, then $\left(m \cdot \varphi\right)\left(x\right) = \varphi^m\left(x\right) = \varphi\left(xm\right)$ for $x \in M$.*

3) The set of all pairs (m, φ) where $m \in M, \varphi \in N^M$, with multiplications operation given by: $\left(m, \varphi\right)\left(m', \psi\right) = \left(mm', \varphi\psi^m\right)$ where $m, m' \in M$ and $\varphi, \psi \in N^M$ is a monoid [1, 8, 9].

Proof.

1) First we will prove that the set N^M froms a monoid shch that for any $\varphi, \psi \in N^M$, let $\varphi\psi$ in N^M be defined for all $m \in M$ by: $\left(\varphi\psi\right)\left(m\right) = \varphi\left(m\right)\psi\left(m\right)$.

a. N^M is non-empty and is closed with respect to multiplication. If $\varphi, \psi \in N^M$, then $\varphi(m), \psi(m) \in N$, for all $m \in M$. Hence $\varphi(m)\psi(m) \in N$. This implies that $(\varphi\psi)(m) \in N$ and so $\varphi\psi \in N^M$.

b. Since multuplication in N is associative, so also is the multuplication in N^M.

c. The identity element in N^M is the map $e : M \longrightarrow N$ given by $e(m) = 1_N$, for all $m \in M$, where 1_N is the identity element of N.

2) Second, we will prove that M is a transformation monoid on N^M in the following was: if $m \in M, \varphi \in N^M$, then $(m \cdot \varphi)(x) = \varphi^m(x) = \varphi(xm)$ for $x \in M$.

Take $\varphi, \psi \in N^M$ and $m, m' \in M$, then

$$\begin{aligned}
((mm') \cdot f)(x) &= f(xmm') \\
(m \cdot (m' \cdot f))(x) &= \left(m \cdot f^{m'}\right)(x) \\
&= \left(f^{m'}\right)^m (x) \\
&= f^{m'}(xm) = f(xmm'). \\
\varphi^{1_M}(x) &= \varphi(x1_M) \\
&= \varphi(x). \\
(\varphi\psi)^m(x) &= \varphi\psi(xm) \\
&= \varphi(xm)\psi(xm) \\
&= \varphi^m(x)\psi^m(x). \\
(\varphi^m)^{m'}(x) &= \varphi^m(xm') \\
&= \varphi(xm'm) \\
&= \varphi^{m'm}(x).
\end{aligned}$$

Then $(\varphi^m)^{m'} = \varphi^{m'm}$.

3) We will prove that $M \times N^M$ is a monoid with multiplication:

$$(m, \varphi)(m', \psi) = (mm', \varphi\psi^m)$$

where $m, m' \in M$ and $\varphi, \psi \in N^M$:

a. Closure property follows from the definition of multiplication.

b. Take $\varphi, \psi, \eta \in N^M$ and $m, m', m" \in G$, then

$$\begin{aligned}
((m, \varphi)(m', \psi))(m'', \eta) &= (mm', \varphi\psi^m)(m'', \eta) \\
&= \left((mm')m'', \varphi\psi^m\eta^{mm'}\right).
\end{aligned}$$

Also we have

$$\begin{aligned}
(m, \varphi)((m', \psi)(m'', \eta)) &= (m, \varphi)\left(m'm'', \psi\eta^{m'}\right) \\
&= \left(m(m'm''), \varphi\left(\psi\eta^{m'}\right)^m\right) \\
&= \left(m(m'm''), \varphi\psi^m\eta^{mm'}\right).
\end{aligned}$$

c. We know that for every $\varphi \in N^M, \varphi^{1_M} = \varphi$, now for every $m \in M$, the map $\varphi \longrightarrow \varphi^m$ is an automorphism of N^M. Also if e is the identity element in N^M, then $e^m = e$. We have $(m, \varphi)(1_M, e) = (m1_M, \varphi e^m) = (m, \varphi e) = (m, \varphi)$. Also $(1_M, e)(m, \varphi) = \left(1_M m, e\varphi^{1_M}\right) = (m, e\varphi) = (m, \varphi)$. Thus identity element exists.

Hence $M \times N^M$ is a monoid with respect to the multiplication defined above. □

Remark 1. *If the monoid M is commutative, then $M \times N^M$ is a monoid with multiplication $(m, \varphi)(m', \psi) = \left(mm', \varphi^{m'}\psi\right)$ where $m, m' \in M$ and $\varphi, \psi \in N^M$.*

Proposition 5. *Let M and N be two monoids, and let M^N denote the set of all functions from the monoid N to the monoid M, then the set $M^N \times N$ is a monoid under the multiplication $(\varphi, n_1)(\psi, n_2) = (\varphi\psi, n_1 n_2)$ where $\varphi\psi \in M^N$ is defined by*

$$\varphi\psi(x) = \varphi(x)\psi(xn_1)$$

for $x, n_1, n_2 \in N$ and $\varphi, \psi \in M^N$.

We call the monoid $M^N \times N$ the wreath product of M and N [1, 8, 9].

Proof. From [6]:

1) We will prove that the multiplication is associative on $M^N \times N$. Let $\varphi, \psi, \eta \in M^N$ and $n_1, n_2, n_3 \in N$ then

$$\begin{aligned} ((\varphi, n_1)(\psi, n_2))(\eta, n_3) &= (\varphi\psi, n_1 n_2)(\eta, n_3) \\ &= ((\varphi\psi)\eta, n_1 n_2 n_3). \end{aligned}$$

And

$$\begin{aligned} (\varphi, n_1)((\psi, n_2)(\eta, n_3)) &= (\varphi, n_1)(n_2 n_3, \psi\eta) \\ &= (\varphi(\psi\eta), n_1 n_2 n_3). \end{aligned}$$

The, we will prove that

$$(\varphi\psi)\eta = \varphi(\psi\eta).$$

Let $x \in N$, then

$$\begin{aligned} ((\varphi\psi)\eta)(x) &= (\varphi\psi)(x)\eta(xn_1 n_2) \\ &= \varphi(x)\psi(xn_1)\eta(xn_1 n_2). \end{aligned}$$

And

$$\begin{aligned} (\varphi(\psi\eta))(x) &= \varphi(x)\psi\eta(xn_1) \\ &= \varphi(x)\psi(xn_1)\eta(xn_1 n_2). \end{aligned}$$

2) Let the map $e : N \longrightarrow M$ given by $e(n) = 1_M$ for all $n \in N$. We have

$$\begin{aligned} (\varphi, n)(e, 1_N) &= (\varphi e, n1_N) \\ &= (\varphi e, n), \end{aligned}$$

where

$$\begin{aligned}(\varphi e)(x) &= \varphi(x) e(xn) \\ &= \varphi(x) 1_M = \varphi(x)\end{aligned}$$

for $n, x \in N$ and $\varphi \in M^N$, then

$$(\varphi, n)(e, 1_N) = (\varphi, n).$$

Also

$$\begin{aligned}(e, 1_N)(\varphi, n) &= (e\varphi, 1_N n) \\ &= (e\varphi, n),\end{aligned}$$

where

$$\begin{aligned}(e\varphi)(x) &= e(x) \varphi(x 1_N) \\ &= 1_M \varphi(x) \\ &= \varphi(x).\end{aligned}$$

Then $(e, 1_N)(\varphi, n) = (\varphi, n)$. The identity element in $M^N \times N$ is $(e, 1_N)$.

□

4 Applications of Products of Monoids on The Monoids of State Machines

In this section, we present the direct product of state machines, the cascade product and wreath product.

Definition 1. *Let $S_1 = (Q_1, \Sigma_1, \delta_1)$ and $S_2 = (Q_2, \Sigma_2, \delta_2)$ be state machines. Suppose that S_1 and S_2 are state machines with the same unput Σ. Connecting them up in this way, will produce a new state machine $S_1 \times S_2 = (Q_1 \times Q_2, \Sigma, \delta_1 \times \delta_2)$ where $(\delta_1 \times \delta_2)((q_1, q_2), \sigma) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma))$ for $\sigma \in \Sigma, (q_1, q_2) \in Q_1 \times Q_2$.*
We call this state machine the restricted direct product of S_1 and S_2 [5, 6].

Example 2. *Let $S_1 = (Q_1, \Sigma_1, \delta_1)$ be state machine where $Q_1 = \{0, 1\}, \Sigma_1 = \{\sigma\}$ and $\delta_1 : Q_1 \times \Sigma_1 \longrightarrow Q_1$ given by:*

δ_1	σ
0	1
1	0

and $S_2 = (Q_2, \Sigma_2, \delta_2)$ given by

$$\begin{aligned}Q_2 &= \{0, 1\}, \\ \Sigma_2 &= \{\sigma\} \\ \delta_2 : Q_2 \times \Sigma_2 &\longrightarrow Q_2\end{aligned}$$

δ_2	σ
0	1
1	1

Then $S_1 \times S_2 = (Q_1 \times Q_2, \Sigma, \delta_1 \times \delta_2)$ where $Q_1 \times Q_2 = \{(0,0), (0,1), (1,0), (1,1)\}$, $\delta_1 \times \delta_2 : (Q_1 \times Q_2) \times \Sigma \longrightarrow (Q_1 \times Q_2)$ given by:

$\delta_1 \times \delta_2$	σ
$(0,0)$	$(1,1)$
$(0,1)$	$(1,1)$
$(1,0)$	$(0,1)$
$(1,1)$	$(0,1)$

Define $\delta_\sigma : Q_1 \longrightarrow Q_1$ *by* $\delta_\sigma(q) = \delta(q,\sigma)$ *for each* $q \in Q_1$. *We have* $\delta_\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $\delta_\sigma \circ \delta_\sigma = Id_{Q_1}$, *Let* $w \in \{\sigma\}^+$ *be a word of length at least* 1 *with symbols from* $\{\sigma\}$. *Suppose that* $w = \sigma^n, n \in \mathbb{N}^*$ *then we define* $(\delta_1)_w : Q_1 \longrightarrow Q_1$ *by* $(\delta_1)_w(q) = \delta_\sigma\delta_\sigma \cdots \delta_\sigma(q)$. *We have*

$$\delta_w = \begin{cases} Id_{Q_1} & if\ n = 2k, k \in \mathbb{N} \\ \delta_\sigma & if\ n = 2k+1, k \in \mathbb{N} \end{cases}$$

$$h_1 : (\{\sigma\}^*, \cdot) \longrightarrow \left(Q_1^{Q_1}, \circ\right)$$

by $h_1(w) = \delta_w$. *Define the relation* $\mathcal{R}_1$ *on* $\{\sigma\}^*$ *by* $w\mathcal{R}_1w'$ *if and only if* $h_1(w) = h_1(w')$. *This is easily verified to be an equivalence relation. Furthermore, it is a congruence relation. The qoutient* $\{\sigma\}^* / \mathcal{R}_1 = \{[\epsilon], [\sigma]\}$.

The monoid of the state machine $(Q_1, \{\sigma\}, \delta_1)$ *is given by:*

$\cdot$	$[\epsilon]$	$[\sigma]$
$[\epsilon]$	$[\epsilon]$	$[\sigma]$
$[\sigma]$	$[\sigma]$	$[\epsilon]$

Define $\delta_2 : Q_2 \longrightarrow Q_2$ *by* $\delta_\sigma(q) = \delta(q,\sigma)$ *for each* $q \in Q_2$. *We have* $\delta_\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, *Let* $w \in \{\sigma\}^+$ *be a word of length at least* 1 *with symbols from* $\{\sigma\}$. *Suppose that* $w = \sigma^n, n \in \mathbb{N}^*$ *then we define* $(\delta_2)_w : Q_2 \longrightarrow Q_2$ *by* $(\delta_2)_w(q) = \delta_\sigma\delta_\sigma \cdots \delta_\sigma(q)$. *We have*

$$\delta_w = \begin{cases} Id_{Q_1} & if\ n = 0 \\ \delta_\sigma & if\ n \in \mathbb{N}^* \end{cases}$$

$$h_2 : (\{\sigma\}^*, \cdot) \longrightarrow \left(Q_2^{Q_2}, \circ\right)$$

by $h_2(w) = \delta_w$. *Define the relation* $\mathcal{R}_2$ *on* $\{\sigma\}^*$ *by* $w\mathcal{R}_2w'$ *if and only if* $h_2(w) = h_2(w')$. *This is easily verified to be an equivalence relation. Furthermore, it is a congruence relation. The qoutient* $\{\sigma\}^* / \mathcal{R}_2 = \{[\epsilon], [\sigma]\}$.

The monoid of the state machine $(Q_2, \{\sigma\}, \delta_2)$ *is given by:*

$\cdot$	$[\epsilon]$	$[\sigma]$
$[\epsilon]$	$[\epsilon]$	$[\sigma]$
$[\sigma]$	$[\sigma]$	$[\sigma]$

Define $(\delta_1 \times \delta_2)_\sigma : Q_1 \times Q_2 \longrightarrow Q_1 \times Q_2$ *by* $(\delta_1 \times \delta_2)_\sigma((q_1,q_2)) = (\delta_1(q_1,\sigma), \delta_2(q_2,\sigma))$ *for each* $(q_1,q_2) \in Q_1 \times Q_2$. *We have*

$$\begin{aligned} (\delta_1 \times \delta_2)_\sigma &= \varphi = \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (1,1) & (1,1) & (0,1) & (0,1) \end{pmatrix}, \\ \varphi^2 &= \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (0,1) & (0,1) & (1,1) & (1,1) \end{pmatrix}, \\ \varphi^3 &= \varphi. \end{aligned}$$

Let $w \in \{\sigma\}^+$ *be a word of length at least* 1 *with symbols from* $\{\sigma\}$. *Suppose that* $w = \sigma^n, n \in \mathbb{N}^*$ *then we define*

$$(\delta_1 \times \delta_2)_w : Q_1 \times Q_2 \longrightarrow Q_1 \times Q_2$$

by

$$(\delta_1 \times \delta_2)_w((q_1,q_2)) = \delta_\sigma\delta_\sigma \cdots \delta_\sigma((q_1,q_2)).$$

We have

$$(\delta_1 \times \delta_2)_w = \begin{cases} Id_{Q_1} & \text{if } n = 0 \\ \varphi & \text{if } n = 2k+1, k \in \mathbb{N} \\ \varphi^2 & \text{if } n = 2k, k \in \mathbb{N}^* \end{cases}$$

$$\psi : (\{\sigma\}^*, \cdot) \longrightarrow \left((Q_1 \times Q_2)^{Q_1 \times Q_2}, \circ\right)$$

by $\psi(w) = (\delta_1 \times \delta_2)_w$.

Define the relation $\mathcal{R}$ *on* $\{\sigma\}^*$ *by* $w\mathcal{R}w'$ *if and only if* $\psi(w) = \psi(w')$. *This is easily verified to be an equivalence relation. Furthermore, it is a congruence relation. The qoutient* $\{\sigma\}^* / \mathcal{R} = \{[\epsilon], [\sigma], [\sigma^2]\}$. *The monoid of the state machine* $(Q_1 \times Q_2, \{\sigma\}, \delta_1 \times \delta_2)$ *is given by:*

$\cdot$	$[\epsilon]$	$[\sigma]$	$[\sigma^2]$
$[\epsilon]$	$[\epsilon]$	$[\sigma]$	$[\sigma^2]$
$[\sigma]$	$[\sigma]$	$[\sigma^2]$	$[\sigma]$
$[\sigma^2]$	$[\sigma^2]$	$[\sigma]$	$[\sigma^2]$

Definition 2. *Let* $S_1 = (Q_1, \Sigma_1, \delta_1)$ *and* $S_2 = (Q_2, \Sigma_2, \delta_2)$ *be state machines. We define*

$$S_1 \times S_2 = (Q_1 \times Q_2, \Sigma_1 \times \Sigma_2, \delta_1 \times \delta_2)$$

where

$$(\delta_1 \times \delta_2)((q_1, q_2), (\sigma_1, \sigma_2)) = (\delta_1(q_1, \sigma_1), \delta_2(q_2, \sigma_2))$$

for

$$(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2, (q_1, q_2) \in Q_1 \times Q_2.$$

We call this state machine the full direct product of S_1 *and* S_2 *[5, 6].*

Example 3. *Let* $S_1 = (Q_1, \Sigma_1, \delta_1)$ *be state machine where* $Q_1 = \{0, 1\}, \Sigma_1 = \{\sigma\}$ *and* $\delta_1 : Q_1 \times \Sigma_1 \longrightarrow Q_1$ *given by:*

δ_1	σ
0	1
1	0

and $S_2 = (Q_2, \Sigma_2, \delta_2)$ *given by*

$$Q_2 = \{0, 1\},$$
$$\Sigma_2 = \{\sigma, \tau\}$$
$$\delta_2 : Q_2 \times \Sigma_2 \longrightarrow Q_2$$

δ_2	σ	τ
0	1	0
1	1	0

Then $S_1 \times S_2 = (Q_1 \times Q_2, \Sigma_1 \times \Sigma_2, \delta_1 \times \delta_2)$ *where* $Q_1 \times Q_2 = \{(0,0), (0,1), (1,0), (1,1)\}$, $\delta_1 \times \delta_2 : (Q_1 \times Q_2) \times \Sigma_1 \times \Sigma_2 \longrightarrow (Q_1 \times Q_2)$ *given by:*

$\delta_1 \times \delta_2$	(σ, σ)	(σ, τ)
$(0,0)$	$(1,1)$	$(1,0)$
$(0,1)$	$(1,1)$	$(1,0)$
$(1,0)$	$(0,1)$	$(0,0)$
$(1,1)$	$(0,1)$	$(0,0)$

Define $\delta_\sigma : Q_1 \longrightarrow Q_1$ *by* $\delta_\sigma(q) = \delta(q,\sigma)$ *for each* $q \in Q_1$*. We have*

$$\delta_\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\delta_\sigma \circ \delta_\sigma = Id_{Q_1}$*, Let* $w \in \{\sigma\}^+$ *be a word of length at least 1 with symbols from* $\{\sigma\}$*. Suppose that* $w = \sigma^n, n \in \mathbb{N}^*$ *then we define* $(\delta_1)_w : Q_1 \longrightarrow Q_1$ *by* $(\delta_1)_w(q) = \delta_\sigma\delta_\sigma \cdots \delta_\sigma(q)$*. We have*

$$\delta_w = \begin{cases} Id_{Q_1} & \text{if } n = 2k, k \in \mathbb{N} \\ \delta_\sigma & \text{if } n = 2k+1, k \in \mathbb{N} \end{cases}$$

$$h_1 : (\{\sigma\}^*, \cdot) \longrightarrow \left(Q_1^{Q_1}, \circ\right)$$

by $h_1(w) = \delta_w$*. Define the relation* $\mathcal{R}_1$ *on* $\{\sigma\}^*$ *by* $w\mathcal{R}_1w'$ *if and only if* $h_1(w) = h_1(w')$*. This is easily verified to be an equivalence relation. Furthermore, it is a congruence relation. The qoutient* $\{\sigma\}^*/\mathcal{R}_1 = \{[\epsilon], [\sigma]\}$*.*
The monoid of the state machine $(Q_1, \{\sigma\}, \delta_1)$ *is given by:*

$\cdot$	$[\epsilon]$	$[\sigma]$
$[\epsilon]$	$[\epsilon]$	$[\sigma]$
$[\sigma]$	$[\sigma]$	$[\epsilon]$

Define $\delta_\sigma : Q_2 \longrightarrow Q_2$ *by* $\delta_\sigma(q) = \delta_2(q,\sigma)$ *for each* $q \in Q_2$*,* $\delta_\tau : Q_2 \longrightarrow Q_2$ *by* $\delta_\tau(q) = \delta_2(q,\tau)$ *for each* $q \in Q_2$*. We have*

$$\begin{aligned} \delta_\sigma &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \\ \delta_\tau &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \delta_\sigma \circ \delta_\tau &= \delta_\sigma \circ \delta_\sigma \\ &= \delta_\sigma, \delta_\tau \circ \delta_\sigma \\ &= \delta_\tau \circ \delta_\tau \\ &= \delta_\tau, \end{aligned}$$

Let $w \in \{\sigma,\tau\}^+$ *be a word of length at least 1 with symbols from* $\{\sigma,\tau\}$*. We have*

$$\delta_w = \begin{cases} Id_{Q_1} & \text{if } w = \epsilon \\ \delta_\sigma & \text{if } w \in \{\sigma,\tau\}^*\sigma \\ \delta_\tau & \text{if } w \in \{\sigma,\tau\}^*\tau \end{cases}$$

$$h_2 : (\{\sigma\}^*, \cdot) \longrightarrow \left(Q_2^{Q_2}, \circ\right)$$

by $h_2(w) = \delta_w$*. Define the relation* $\mathcal{R}_2$ *on* $\{\sigma,\tau\}^*$ *by* $w\mathcal{R}_2w'$ *if and only if* $h_2(w) = h_2(w')$*. This is easily verified to be an equivalence relation. Furthermore, it is a congruence relation. The qoutient* $\{\sigma,\tau\}^*/\mathcal{R}_2 = \{[\epsilon], [\sigma], [\tau]\}$*.*

The monoid of the state machine $(Q_2, \{\sigma,\tau\}, \delta_2)$ *is given by:*

$\cdot$	$[\epsilon]$	$[\sigma]$	$[\tau]$
$[\epsilon]$	$[\epsilon]$	$[\sigma]$	$[\tau]$
$[\sigma]$	$[\sigma]$	$[\sigma]$	$[\tau]$
$[\tau]$	$[\tau]$	$[\sigma]$	$[\tau]$

.

Define $(\delta_1 \times \delta_2)_{(\sigma,\sigma)} : Q_1 \times Q_2 \longrightarrow Q_1 \times Q_2$ *by* $(\delta_1 \times \delta_2)_{(\sigma,\sigma)}((q_1, q_2)) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma))$ *for each* $(q_1, q_2) \in Q_1 \times Q_2$. *We have*

$$\begin{aligned}(\delta_1 \times \delta_2)_{(\sigma,\sigma)} = \eta &= \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (1,1) & (1,1) & (0,1) & (0,1) \end{pmatrix}, \\ \eta^2 &= \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (0,1) & (0,1) & (1,1) & (1,1) \end{pmatrix}, \\ \eta^3 &= \eta.\end{aligned}$$

And $(\delta_1 \times \delta_2)_{(\sigma,\tau)} : Q_1 \times Q_2 \longrightarrow Q_1 \times Q_2$ *by* $(\delta_1 \times \delta_2)_{(\sigma,\tau)}((q_1, q_2)) = (\delta_1(q_1, \sigma), \delta_2(q_2, \tau))$ *for each* $(q_1, q_2) \in Q_1 \times Q_2$. *We have*

$$\begin{aligned}(\delta_1 \times \delta_2)_{(\sigma,\tau)} = \mu &= \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (1,0) & (1,0) & (0,0) & (0,0) \end{pmatrix}, \\ \mu^2 &= \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (0,0) & (0,0) & (1,0) & (1,0) \end{pmatrix}, \\ \mu^3 &= \mu.\end{aligned}$$

The monoid of the state machine $S_1 \times S_2 = (Q_1 \times Q_2, \Sigma_1 \times \Sigma_2, \delta_1 \times \delta_2)$ *is given by:*

$\cdot$	$[\epsilon]$	$[(\sigma,\sigma)]$	$[(\sigma,\sigma)^2]$	$[(\sigma,\tau)]$	$[(\sigma,\tau)^2]$
$[\epsilon]$	$[\epsilon]$	$[(\sigma,\sigma)]$	$[(\sigma,\sigma)^2]$	$[(\sigma,\tau)]$	$[(\sigma,\tau)^2]$
$[(\sigma,\sigma)]$	$[(\sigma,\sigma)]$	$[(\sigma,\sigma)^2]$	$[(\sigma,\sigma)]$	$[(\sigma,\tau)^2]$	$[(\sigma,\tau)]$
$[(\sigma,\sigma)^2]$	$[(\sigma,\sigma)^2]$	$[(\sigma,\sigma)]$	$[(\sigma,\sigma)^2]$	$[(\sigma,\tau)]$	$[(\sigma,\tau)^2]$
$[(\sigma,\tau)]$	$[(\sigma,\tau)]$	$[(\sigma,\sigma)^2]$	$[(\sigma,\sigma)]$	$[(\sigma,\tau)^2]$	$[(\sigma,\tau)]$
$[(\sigma,\tau)^2]$	$[(\sigma,\tau)^2]$	$[(\sigma,\sigma)]$	$[(\sigma,\sigma)^2]$	$[(\sigma,\tau)]$	$[(\sigma,\tau)^2]$

Definition 3. *Let* $S_1 = (Q_1, \Sigma_1, \delta_1)$ *and* $S_2 = (Q_2, \Sigma_2, \delta_2)$ *be state machines. We define the cascade product of* S_1 *and* S_2 *with respect to* $\omega : Q_2 \times \Sigma_2 \longrightarrow \Sigma_1$ *by*

$$S_1 \omega S_2 = (Q_1 \times Q_2, \Sigma_2, \delta^\omega)$$

where

$$\delta^\omega((q_1, q_2), \sigma_2) = (\delta_1(q_1, \omega(q_2, \sigma_2)), \delta_2(q_2, \sigma_2))$$

for $\sigma_2 \in \Sigma_2, (q_1, q_2) \in Q_1 \times Q_2$ *[5, 6].*

Example 4. *Let* $S_1 = (Q_1, \Sigma_1, \delta_1)$ *be state machine where* $Q_1 = \{0, 1\}, \Sigma_1 = \{\sigma, \tau\}$ *and* $\delta_1 : Q_1 \times \Sigma_1 \longrightarrow Q_1$ *given by:*

δ_1	σ	τ
0	1	0
1	1	0

and $S_2 = (Q_2, \Sigma_2, \delta_2)$ *given by* $Q_2 = \{0, 1\}$, $\Sigma_2 = \{\sigma\}$ *and* $\delta_2 : Q_2 \times \Sigma_2 \longrightarrow Q_2$

δ_2	σ
0	1
1	0

. *Define a mapping* $\omega : Q_2 \times \Sigma_2 \longrightarrow \Sigma_1$ *by* $\omega(0, \sigma) = \sigma, \omega(1, \sigma) = \tau$.

The cascade product $S_1 \omega S_2 = (Q_1 \times Q_2, \Sigma_2, \delta^\omega)$ *where given by*

δ^ω	σ
$(0,0)$	$(1,1)$
$(0,1)$	$(0,0)$
$(1,0)$	$(1,1)$
$(1,1)$	$(0,0)$

Define $(\delta^\omega)_\sigma : Q_1 \times Q_2 \longrightarrow Q_1 \times Q_2$ *by* $(\delta^\omega)_\sigma ((q_1, q_2)) = (\delta_1 (q_1, \omega (q_2, \sigma)), \delta_2 (q_2, \sigma))$ *for each* $(q_1, q_2) \in Q_1 \times Q_2$*. We have*

$$
\begin{aligned}
(\delta^\omega)_\sigma = \phi &= \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (1,1) & (0,0) & (1,1) & (0,0) \end{pmatrix}, \\
(\delta^\omega)_{\sigma\sigma} = \phi^2 &= \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (0,0) & (1,1) & (0,0) & (1,1) \end{pmatrix}.
\end{aligned}
$$

The monoid of the state machine $S_1 \omega S_2$ *is given by*

$\cdot$	$[\epsilon]$	$[\sigma]$	$[\sigma^2]$
$[\epsilon]$	$[\epsilon]$	$[\sigma]$	$[\sigma^2]$
$[\sigma]$	$[\sigma]$	$[\sigma^2]$	$[\sigma]$
$[\sigma^2]$	$[\sigma^2]$	$[\sigma]$	$[\sigma^2]$

Definition 4. *Let* $S_1 = (Q_1, \Sigma_1, \delta_1)$ *and* $S_2 = (Q_2, \Sigma_2, \delta_2)$ *be state machines. We define the wreath product of* S_1 *and* S_2 *by* $S_1 W S_2 = \left(Q_1 \times Q_2, \Sigma_1^{Q_2} \times \Sigma_2, \delta^W\right)$ *where* $\delta^W ((q_1, q_2), (f, \sigma_2)) = (\delta_1 (q_1, f(q_2)), \delta_2 (q_2, \sigma_2))$ *for* $\sigma_2 \in \Sigma_2, f \in \Sigma_1^{Q_2}, (q_1, q_2) \in Q_1 \times Q_2$ *[5, 6, 7].*

Example 5. *Let* $S_1 = (Q_1, \Sigma_1, \delta_1)$ *be state machine where* $Q_1 = \{0, 1\}, \Sigma_1 = \{\sigma, \tau\}$ *and* $\delta_1 : Q_1 \times \Sigma_1 \longrightarrow Q_1$ *given by:*

δ_1	σ	τ
0	1	0
1	1	0

and $S_2 = (Q_2, \Sigma_2, \delta_2)$ *given by* $Q_2 = \{0, 1\}$, $\Sigma_2 = \{\sigma\}$ *and* $\delta_2 : Q_2 \times \Sigma_2 \longrightarrow Q_2$

δ_2	σ
0	1
1	0

. Denote the four elements of $\Sigma_1^{Q_2}$ *by* f_1, f_2, f_3, f_4 *where*

$$
\begin{aligned}
f_1(0) &= f_1(0) = \sigma. \\
f_2(0) &= \sigma, f_2(1) = \tau. \\
f_3(0) &= \tau, f_3(1) = \sigma. \\
f_4(0) &= f_4(1) = \tau.
\end{aligned}
$$

Then the state machine $S_1 W S_2$ *has the table*

δ^W	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(f_1, σ)	$(1,1)$	$(1,0)$	$(1,1)$	$(1,0)$
(f_2, σ)	$(1,1)$	$(0,0)$	$(1,1)$	$(0,0)$
(f_2, σ)	$(0,1)$	$(1,0)$	$(0,1)$	$(1,0)$
(f_4, σ)	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$

Define $\left(\delta^{W}\right)_{(f_1,\sigma)} : Q_1 \times Q_2 \longrightarrow Q_1 \times Q_2$ *by* $\left(\delta^{W}\right)_{(f_1,\sigma)}((q_1,q_2)) = (\delta_1(q_1, f_1(q_2)), \delta_2(q_2,\sigma))$ *for each* $(q_1,q_2) \in Q_1 \times Q_2$*. We have*

$$\begin{aligned}
\left(\delta^{W}\right)_{(f_1,\sigma)} &= \alpha = \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (1,1) & (1,0) & (1,1) & (1,0) \end{pmatrix}, \\
\left(\delta^{W}\right)_{(f_2,\sigma)} &= \beta = \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (1,1) & (0,0) & (1,1) & (0,0) \end{pmatrix}, \\
\left(\delta^{W}\right)_{(f_3,\sigma)} &= \gamma = \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (0,1) & (1,0) & (0,1) & (1,0) \end{pmatrix} \\
\left(\delta^{W}\right)_{(f_4,\sigma)} &= \lambda = \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (0,1) & (0,0) & (0,1) & (0,0) \end{pmatrix}.
\end{aligned}$$

5 Conclusion

In this paper, we give a specific transformation monoid, after that, we give the monoids of state machines associate with the direct product, the cascade product and wreath product of state machines.

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